UNIT-3

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1° * **Inner-Product Space:-** let X be a linear space over the scalar field C of complex numbers. An inner-product on X is a function $(\cdot,\cdot): X \times X \to C$ which satisfies the following conditions:

(i) $(x,y) = \overline{(y,x)}$ for $x,y \in X$.

(ii) $(\lambda x + \mu y, z) = \lambda(x+z) + \mu(y+z)$
$\lambda(x,z) + \mu(y,z)$ for $\lambda,\mu \in C$, $x,y,z \in X$

(iii) $(x,x) \geq 0$; $(x,x) = 0$ if and only if $x = 0$. $\forall x \in X$

OR:- let X be a linear space over the field K then an inner product space is a complex valued function
$\langle \cdot,\cdot \rangle : X \times X \to K$ which satisfy the following conditions:-

(i) $\langle x,x \rangle \geq 0 \quad \forall x \in X$
and $\langle x,x \rangle = 0$ iff $x = 0$.

(ii) $\langle x,y \rangle = \overline{\langle y,x \rangle} \quad \forall x,y \in X$

(iii) $\langle \alpha x, y \rangle = \alpha \langle x,y \rangle \quad \forall x,y \in X$ and $\alpha \in K$

(iv) $\langle x+y, z \rangle = \langle x,z \rangle + \langle y,z \rangle \quad \forall x,y,z \in X$.

The value $\langle x,y \rangle$ is called the inner product of x and y and the pair $\langle x, \langle \cdot,\cdot \rangle \rangle$ is called inner product space or pre hilbert space.

Ex:-
① R^n is real inner product space with the inner product
$\langle x,y \rangle = \sum_{k=1}^{n} x_k y_k$

② C^n is an complex inner product space with the inner product.
$\langle x,y \rangle = \sum_{k=1}^{n} x_k \overline{y_k}$

* **Inner Product Space:-** An inner product space is a vector space X with an inner product defined on X.

* **Hilbert Space:-** An inner-product space is called a ~~Hilbert~~ Hilbert space if the normed space induced by the inner-product is a Banach space.

Remark:- By the definition of inner product space we have
(i) $\langle \lambda x + \mu y, z \rangle = \lambda \langle x,z \rangle + \mu \langle y,z \rangle$
(ii) $\langle x, \mu y \rangle = \bar{\mu} \langle x,y \rangle$
(iii) $\langle x, \lambda y + \mu z \rangle = \bar{\lambda} \langle x,y \rangle + \bar{\mu} \langle x,z \rangle$.

Note:- (i) An inner product on X define a norm on X given by.
$\|\cdot\| = \sqrt{\langle \cdot,\cdot \rangle}$

(ii) A metric on X given by.
$d(x,y) = \|x-y\|$
$= \sqrt{\langle x-y, x-y \rangle}$

Hence, inner product space are normed space and Hilbert space are Banach space.

Cauchy-Schwarz inequality :-

Statement :- If X is an inner-product space and $x, y \in X$, then
$|\langle x, y\rangle| \leq \|x\| \|y\|$.

Proof :- if $y = 0$ the inequality holds. So, let $y \neq 0$ and $\lambda \in \mathbb{C}$ then we have,

$$0 \leq \|x - \lambda y\|^2 = \langle x - \lambda y, x - \lambda y\rangle$$
$$= \langle x, x\rangle - \bar{\lambda}\langle x, y\rangle - \lambda\langle y, x\rangle + \lambda\bar{\lambda}\langle y, y\rangle$$
$$= \|x\|^2 - \bar{\lambda}\langle x, y\rangle - \lambda\langle y, x\rangle + \lambda\bar{\lambda}\|y\|^2 \quad \text{---(1)}$$

put $\lambda = \frac{\langle x, y\rangle}{\|y\|^2}$ in eqn (1) we get;

$$0 \leq \|x\|^2 - \frac{\overline{\langle x, y\rangle}}{\|y\|^2}\langle x, y\rangle - \frac{\langle x, y\rangle}{\|y\|^2}\langle y, x\rangle + \frac{\langle x, y\rangle\overline{\langle x, y\rangle}}{\|y\|^4}\|y\|^2$$

$$\Rightarrow 0 \leq \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2} - \frac{|\langle x, y\rangle|^2}{\|y\|^2} + \frac{|\langle x, y\rangle|^2}{\|y\|^2}$$

$$\Rightarrow 0 \leq \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2}$$

$$\Rightarrow 0 \leq \|x\|^2\|y\|^2 - |\langle x, y\rangle|^2$$

$$\Rightarrow |\langle x, y\rangle|^2 \leq \|x\|^2\|y\|^2$$

$$\boxed{|\langle x, y\rangle| \leq \|x\|\|y\|}$$ Hence proved.

Theorem :-

Theorem :- If X is an inner product space then $\|\cdot\|$ defined a norm on X.

Proof :- To prove that $\|\cdot\|$ defines a norm on X for $x \in X$ we have :

(i) $\|x\| \geq 0$,
we have. $\langle x, x\rangle \geq 0$
$\Rightarrow \|x\|^2 \geq 0$
$\Rightarrow \|x\| \geq 0$

(ii) $\|x + y\| \leq \|x\| + \|y\|$

$\because \|x + y\|^2 = \langle x + y, x + y\rangle$
$= \langle x, (x + y)\rangle + \langle y, (x + y)\rangle$
$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$
$= \|x\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \|y\|^2$
$\leq \|x\|^2 + 2\,\mathrm{Re}\langle x, y\rangle + \|y\|^2$
$\because 2\,\mathrm{Re}\langle x, y\rangle \leq 2|\langle x, y\rangle|$

$\leq \|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2$

$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$

$= (\|x\| + \|y\|)^2$

$\Rightarrow \|x+y\| \leq \|x\| + \|y\|$

(iii) $\|x\| = 0 \Rightarrow \sqrt{\langle x,x\rangle} = 0$

$\Rightarrow \langle x,x\rangle = 0$

$\Rightarrow x = 0$

(iv) $\|\lambda x\| = |\lambda|\|x\|$

$\because \|\lambda x\|^2 = \langle \lambda x, \lambda x\rangle$

$= \lambda\bar{\lambda}\langle x,x\rangle$

$= |\lambda|^2\|x\|^2$

$\|\lambda x\| = |\lambda|\|x\|$.

Hence $\|\cdot\|$ defined a norm on X.

* **Theorem:-** In an inner product space the inner product is continuous.

Proof:- let X be an inner product space and let $\{x_n\}$ and $\{y_n\}$ be a sequence in X such that

$\lim_{n\to\infty} x_n = x$ or $x_n \to x$

and $\lim_{n\to\infty} y_n = y$ or $y_n \to y$

Then,

$|\langle x_n, y_n\rangle - \langle x,y\rangle| = |\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x,y\rangle|$

$= |\langle x_n, y_n - y\rangle + \langle x_n - x, y\rangle|$

$\leq |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle|$

$\leq \|x_n\|\|y_n - y\| + \|x_n - x\|\|y\|$

(∵ by cauchy swarz inequality)

since sequence $\{x_n\}$ is convergent so. $\|x_n\|$ is bounded.

similarly $\|y_n\|$ is bounded.

Hence,

$\lim_{n\to\infty} \langle x_n, y_n\rangle = \langle x,y\rangle$

Hence, the inner product space is continuous.

* **Parallelogram law :-**

Statement:- If x and y are any two elements of an inner-product space X, then

$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

Proof:- since X is an inner-product space, we have,

if $x, y \in X$ then

$\|x+y\|^2 = \langle x+y, x+y\rangle$

$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$

$= \|x\|^2 + \langle x,y\rangle + \overline{\langle x,y\rangle} + \|y\|^2$

$= \|x\|^2 + 2\,\mathrm{Re}\langle x,y\rangle + \|y\|^2$ — (1)

and

$\|x-y\|^2 = \langle x-y, x-y\rangle$

$= \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle$

$= \|x\|^2 - \langle x,y\rangle - \overline{\langle x,y\rangle} + \|y\|^2$

$= \|x\|^2 - 2\,\mathrm{Re}\langle x,y\rangle + \|y\|^2$ — (2)

adding eqn (1) and (2) we get,

$\|x+y\|^2 + \|x-y\|^2 = \|x\|^2 + 2\,\mathrm{Re}\langle x,y\rangle + \|y\|^2 + \|x\|^2 - 2\,\mathrm{Re}\langle x,y\rangle + \|y\|^2$

$\boxed{\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2}$ Hence Proved.

or $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

* Polarization identity:

Statement:- If x and y are in an inner product space and $x, y \in X$ then

$\langle x,y\rangle = \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right]$

Proof:- we have

$\|x+y\|^2 = \langle x+y, x+y\rangle$

$= \|x\|^2 + \|y\|^2 + \langle x,y\rangle + \langle y,x\rangle$ — (1)

$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - \langle x,y\rangle - \langle y,x\rangle$ — (2)

$\|x+iy\|^2 = \langle x+iy, x+iy\rangle$

$= \langle x,x\rangle + \langle x,iy\rangle + i\langle y,x\rangle + i^2\langle y,y\rangle$

$= \|x\|^2 + \langle x,iy\rangle + i\langle y,x\rangle - \|y\|^2$ — (3)

and

$\|x-iy\|^2 = \langle x-iy, x-iy\rangle$

$= \langle x,x\rangle - \langle x,iy\rangle - i\langle y,x\rangle + i^2\langle y,y\rangle$

$= \|x\|^2 - \langle x,iy\rangle - i\langle y,x\rangle + \|y\|^2$ — (4)

subtracting eqn (2) and (3) we get

$\|x+y\|^2 - \|x-y\|^2 = \langle x,y\rangle + \langle y,x\rangle + \langle x,y\rangle + \langle y,x\rangle$

$= 2\langle x,y\rangle + 2\langle y,x\rangle$ — (5)

Also from eqn (3) and (4) we have,

$i\|x+iy\|^2 - i\|x-iy\|^2 = i\|x\|^2 + i\langle x,iy\rangle + \langle y,x\rangle - i\|y\|^2 + i\langle x,iy\rangle - \|x\|^2 + i\langle y,x\rangle + i\|y\|^2$

$= 2i\langle x,iy\rangle - \langle y,x\rangle + \langle y,x\rangle$

$= 2\langle x,y\rangle - 2\langle y,x\rangle$ — (6)

Now adding eqn (5) and (6) we get,

$$\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 = 4\langle x,y\rangle$$

$$\langle x,y\rangle = \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right]$$

Hence Proved

* **Uniformly convex:-** let X be an nls and suppose that for all $\varepsilon > 0$ and $x, y \in X$ s.t.

$$\|x\| = \|y\| = 1$$

and $\|x-y\| \geq \varepsilon$

imply that

$$\left\|\frac{1}{2}(x+y)\right\| \leq 1-\delta.$$

where $\delta = \delta(\varepsilon)$ i.e. δ depends on ε but independ of x and y and $0 < \delta < 1$ Then X is said to be uniformly convex.

* **Theorem:-** Every inner-product space X is uniformly convex.

Proof:- let $x, y \in X$, $\|x\| = 1$, $\|y\| = 1$ and $\|x-y\| \geq \varepsilon$.

By parallelogram law we have,

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

$$\|x+y\|^2 = 2\|x\|^2 + 2\|y\|^2 - \|x-y\|^2$$

Dividing both side by 4 we get,

$$\left\|\frac{x+y}{2}\right\|^2 = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{4}\|x-y\|^2$$

$$\leq \frac{1}{2} + \frac{1}{2} - \frac{1}{4}\varepsilon^2$$

$$= 1 - \left(\frac{\varepsilon}{2}\right)^2$$

Therefore, we can find a number δ, $0 < \delta < 1$ such that,

$$\left\|\frac{1}{2}(x+y)\right\| \leq 1-\delta$$

where $(1-\delta)^2 = 1 - \frac{\varepsilon^2}{4}$

Hence, the inner product space X is uniformly convex.

* **Orthonormal (orthogonality):-**

Two element x and y in an inner product space X it is said to be orthogonal if $\langle x,y\rangle = 0$ and it is denoted by $x \perp y$.

A subset S of X is said to be orthogonal if $x \perp y$, $\forall\, x, y \in S$ with $x \neq y$. S is said to be orthonormal if S is orthogonal and $\|x\| = k$ $\forall\, x \in S$.

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09) * **Pythagorean Theorem**

Statement:- If $S = \{x_1, x_2, \ldots, x_n\}$ is an orthogonal subset of an inner product space X, then

$$\left\| \sum_{i=1}^{n} x_i \right\|^2 = \sum_{i=1}^{n} \|x_i\|^2.$$

Proof:- we have,

$$\left\| \sum_{i=1}^{n} x_i \right\|^2 = \left(\sum_{i=1}^{n} x_i, \sum_{j=1}^{n} x_j \right)$$

$$= \sum_{i=1}^{n} \langle x_i, x_i \rangle + \sum_{\substack{i,j=1 \\ i \neq j}} \langle x_i, x_j \rangle$$

$$= \sum_{i=1}^{n} \langle x_i, x_i \rangle + 0 \quad \because \langle x_i, x_j \rangle = 0 \text{ if } x_j \neq x_i$$

$$\left\| \sum_{i=1}^{n} x_i \right\|^2 = \sum_{i=1}^{n} \|x_i\|^2$$

Hence Proved.

* **Definition (complete):-** If X is a non-zero inner product space then the family of all its orthonormal set is non-empty. This family is clearly a partially ordered set with respect to set inclusion as the partial order.

An orthonormal set $\{e_i\}$ in X is said to be complete if it is maximal in the partially ordered set. i.e. if it is impossible to adjoin the vector e to set $\{e_i\}$ s.t. set $\{e_i, e_j\}$ is an orthonormal set which properly contain $\{e_i\}$

Bessel's inequality:-

Statement:- If $\{e_1, e_2, \ldots, e_n\}$ is an finite orthonormal set in an inner product space X and x is any element of X then,

$$\sum_{i=1}^{n} |\langle x, e_i \rangle|^2 \leq \|x\|^2$$

and $x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i \perp e_j \quad (j = 1, 2, \ldots, n)$

Proof:- we have,

$$0 \leq \left\| x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i \right\|^2$$

$$= \left(x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i, \; x - \sum_{j=1}^{n} \langle x, e_j \rangle e_j \right)$$

$$= \langle x, x \rangle - \sum_{j=1}^{n} \overline{\langle x, e_j \rangle} \langle x, e_j \rangle - \sum_{i=1}^{n} \langle x, e_i \rangle \langle e_i, x \rangle + \sum_{i=1}^{n} \sum_{j=1}^{n} \langle x, e_i \rangle \langle x, e_j \rangle \langle e_i, e_j \rangle .$$

$$= \langle x,x\rangle - \sum_{j=1}^{n} \langle x,e_j\rangle \langle x,e_j\rangle + \sum_{i=1}^{n} \langle x,e_i\rangle \overline{\langle x,e_i\rangle}$$
$$+ \sum_{j=1}^{n}\sum_{i=1}^{n} \langle x,e_i\rangle \langle \overline{x,e_j}\rangle \langle e_i,e_j\rangle$$

$$= \langle x,x\rangle - \sum_{j=1}^{n} \overline{\langle x,e_j\rangle}\langle x,e_j\rangle - \sum_{i=1}^{n} \langle x,e_i\rangle \overline{\langle x,e_i\rangle}$$
$$+ \sum_{i=1}^{n} \langle x,e_i\rangle \langle \overline{x,e_j}\rangle \quad \{\because \|e_i\|^2 = 1\}$$

$$= \langle x,x\rangle - \sum_{i=1}^{n} \langle x,e_i\rangle \overline{\langle x,e_i\rangle}$$

$$= \|x\|^2 - \sum_{i=1}^{n} |\langle x,e_i\rangle|^2$$

$$\Rightarrow \sum_{i=1}^{n} |\langle x,e_i\rangle|^2 \le \|x\|^2$$

$$\Rightarrow \sum_{i=1}^{n} |\langle x,e_i\rangle| \le \|x\|$$

Further,

$$\langle x - \sum_{i=1}^{n} \langle x,e_i\rangle \langle e_i,e_j\rangle$$

$$= \langle x,e_j\rangle - \sum_{i=1}^{n} \langle x,e_i\rangle \langle e_i,e_j\rangle$$

$$= \langle x,e_j\rangle - \sum_{i=1}^{n} \langle x,e_i\rangle \langle e_i,e_j\rangle$$

$$= \langle x,e_j\rangle - \langle x,e_j\rangle$$

$$= 0$$

which implies that

$$x - \sum_{i=1}^{n} \langle x,e_i\rangle e_i \perp e_j \quad (j = 1,2,\cdots,n)$$

This complete the proof.

(Parseval's identity)

Theorem:- let $\{e_i\}$ be a non-empty arbitrary orthonormal set in an inner product x. Then the following four condition are equivalent:

(i) $\{e_i\}$ is complete.

(ii) $x \perp \{e_i\} \Rightarrow x = 0$.

(iii) $x \in X \Rightarrow x = \sum_{i=1}^{n} \langle x,e_i\rangle e_i$

(iv) $x \in X \Rightarrow \|x\|^2 = \sum |\langle x,e_i\rangle|^2$ (parseval's identity).

Proof:- (i) $\Rightarrow$ (ii) suppose that $\{e_i\}$ is complete and assume that (ii) is false then there exist $x \neq 0$ such that $x \perp \{e_i\}$.

Define,

$$e = \frac{x}{\|x\|}$$

Then $\langle e_i, e\rangle = \left\langle e_i, \frac{x}{\|x\|}\right\rangle$.

$$= \frac{1}{\|x\|}\langle e_i, x\rangle = 0.$$

$\Rightarrow \langle e_i, e \rangle = 0.$

$\Rightarrow e_i$ and e is orthogonal and also we have $\|e\| = 1$.

which contradicts the completeness of $\{e_i\}$

Hence $x \perp \{e_i\}$

$\Rightarrow x = 0.$

(ii) $\Rightarrow$ (iii) let $x \perp \{e_i\}$

$\Rightarrow x = 0$ and suppose that $x \in X$, then,

$$\langle x - \sum \langle x, e_i \rangle e_i, e_j \rangle$$
$$= \langle x, e_j \rangle - [\sum \langle x, e_i \rangle e_j \rangle$$
$$= \langle x, e_j \rangle - \langle x, e_j \rangle$$
$$= 0$$

$$\Rightarrow \langle x - \sum_{i=1}^{n} \langle x, e_j \rangle, e_i, e_j \rangle = 0$$

$$\Rightarrow \left(\langle x - \sum_{i=1}^{n} \langle x, e_j \rangle e_i\right) \perp \{e_j\}$$

$$\Rightarrow \langle x - \sum_{i=1}^{n} \langle x, e_j \rangle e_i \rangle = 0$$

$$\boxed{x = \sum_{i=1}^{n} \langle x, e_j \rangle e_i}$$

(iii) $\Rightarrow$ (iv) suppose that,

$$x = \sum \langle x, e_i \rangle e_i$$

Now,

$$\|x\|^2 = \langle x, x \rangle$$
$$= \langle \sum \langle x, e_i \rangle e_i, \sum \langle x, e_i \rangle e_i \rangle$$
$$= \sum \langle x, e_i \rangle \overline{\langle x, e_i \rangle} \langle e_i, e_i \rangle$$
$$= \sum |\langle x, e_i \rangle|^2 \|e_i\|^2$$
$$= \sum |\langle x, e_i \rangle|^2$$
$$\|x\|^2 = \sum |\langle x, e_i \rangle|^2$$
$$\Rightarrow \|x\|^2 = \sum |\langle x, e_i \rangle|^2$$

(iv) $\Rightarrow$ (i) suppose that,

$$\|x\|^2 = \sum |\langle x, e_i \rangle|^2 \quad \forall x \in X$$

To prove that $\{e_i\}$ is complete suppose if possible $\{e_i\}$ is not complete then,

$$\{e_i\} \subsetneq \{e, e_i\}$$

where e is orthonormal to e_i i.e. $e \perp \{e_i\}$ and $\|e\| = 1$.

$\therefore e \perp \{e_i\}$ then $e = 0$.

Then by eqn (1)

$$\|e\|^2 = \sum |\langle e, e_i \rangle|^2$$

$$\|e\|^2 = 0$$

which is a contradiction hence $\{e_i\}$ is complete.

Riesz-Fischer Theorem

Statement:- Let X be a separable infinite-dimensional complex Hilbert space. Then X is isometrically isomorphic to l_2.

Proof:- since X is separable Hilbert space then $\exists$ a countable complete orthonormal set $\{x_1, x_2, \ldots, x_n\}$ is complete, Therefore by Parseval's identity, we have for any x.

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2$$

$$= \sum_{n=1}^{\infty} |\alpha_n|^2 < \infty.$$

Now, define a map $T: X \to l_2$ by

$$T(x) = \{\alpha_n\}$$

clearly T is linear. In order to prove that T is isomorphic we have to show that

(i) T is one-one.
(ii) T is onto.

(i) **T is one-one:-** since T is linear, If enough to show that,

$\ker T = \{0\}$.

suppose $x \in \ker T$ then,

$\Rightarrow T(x) = 0$

$\Rightarrow \{\alpha_n\} = 0$

$\Rightarrow \alpha_n = 0 \quad \forall n.$

$\Rightarrow \langle x, x_n \rangle = 0 \quad \forall n.$

$\Rightarrow x = 0.$

Hence $\ker T = \{0\}$

(ii) **T is onto:-** consider an arbitrary element $\{\beta_n\} \in l_2$ since

$$\sum |\beta_n|^2 < \infty.$$

$\therefore \sum_{n=1}^{\infty} \beta_n x_n$ is convergent to sum and $x \in X$ and furthermore,

$\beta_n = \langle x, x_n \rangle$

Thus T is onto.

Finally,

$$\|Tx\|^2 = \sum_{n=1}^{\infty} |\alpha_n|^2 = \|x\|^2.$$

This show that T is an isomorphic and Hence this completes the proof of theorem.

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Theorem:- If K is non-empty closed and convex set in a Hilbert space X. Then ∃ a unique vector in K of smallest norm.

Proof:- Let X be a hilbert space and let K be a closed and convex set in X. Then we have to show that ∃ a unique vector in K of smallest norm d.

let,

$$d = \inf\{\|x\| : x \in X\}$$

then ∃ a sequence $\{x_n\}$ in K s.t.

$$\lim_{n\to\infty} \|x_n\| = d \quad \text{—(1)}$$

Now, by parallelogram law,

$$\left\|\frac{x_n}{2} + \frac{x_m}{2}\right\|^2 + \left\|\frac{x_n}{2} - \frac{x_m}{2}\right\|^2 = 2\left\|\frac{x_n}{2}\right\|^2 + 2\left\|\frac{x_m}{2}\right\|^2$$

$\forall\, x_n, x_m \in K.$ —(2)

Now, since K is convex therefore,

$$\alpha x + (1-\alpha) y \in X \quad \forall\, x, y \in K$$

put $\alpha = \frac{1}{2}$, $x = \frac{x_n}{2}$ and $y = \frac{x_m}{2}$ we get

$$\frac{1}{2}\left(\frac{x_n}{2}\right) + \left(1 - \frac{x_n}{2}\right)\frac{x_m}{2} \in X$$

$$\Rightarrow \frac{x_n}{4} + \left(1 - \frac{x_n}{2}\right)\frac{x_m}{2} \in X$$

$$\Rightarrow \frac{x_n}{4} + \frac{x_m}{2} - \frac{x_n x_m}{4} \in X \qquad \forall\, x_n, x_m \in K.$$

in above eqn we get

$$\frac{x_n}{2} + \frac{x_m}{2} \in K.$$

and Hence, by definition of d

$$\left\|\frac{x_n}{2} + \frac{x_m}{2}\right\| \geq d \quad \text{—(3)}$$

Now, from eqn (2)

$$\Rightarrow \left\|\frac{x_n}{2} + \frac{x_m}{2}\right\|^2 + \left\|\frac{x_n}{2} - \frac{x_m}{2}\right\|^2 = 2\left\|\frac{x_n}{2}\right\|^2 + 2\left\|\frac{x_m}{2}\right\|^2$$

$$\Rightarrow \left\|\frac{x_n}{2} - \frac{x_m}{2}\right\|^2 = 2\left\|\frac{x_n}{2}\right\|^2 + 2\left\|\frac{x_m}{2}\right\|^2 - \left\|\frac{x_n}{2} + \frac{x_m}{2}\right\|^2$$

$$\Rightarrow \|x_n - x_m\|^2 = 2\|x_n\|^2 + 2\|x_m\|^2 - 4\left\|\frac{x_n}{2} + \frac{x_m}{2}\right\|^2$$

$$\Rightarrow \|x_n - x_m\|^2 = 2\|x_n\|^2 + 2\|x_m\|^2 - 4d^2 \quad \{\text{by (3)}\}$$

$$\Rightarrow \|x_n - x_m\|^2 \leq 0 \quad \{\because \|x_n\| = d,\ \|x_m\| = d\}$$

$$\Rightarrow \|x_n - x_m\|^2 = 0$$

$\Rightarrow \sup_{m,n} \|x_n - x_m\| = 0$

$\Rightarrow \lim_{n\to\infty} \|x_n - x_m\| = 0$

$\{x_n\}$ is a cauchy sequence in K since X is complete and K is a closed subset of X.

Thus K is also complete so $\exists$ a element $x \in X$ s.t.

$$\lim_{n\to\infty} x_n = x$$

then, $\|x\| = \|\lim_{n\to\infty} x_n\| = d$

$\Rightarrow \boxed{\|x\| = d}$

Hence, x is a vector in K with smallest norm d.

Uniqueness of d:- Now we have to show that, let $x, y \in K$ s.t. $\|x\| = d$ and $\|y\| = d$. Then we have to show that $x = y$.

again by parallelogram law

$$\Rightarrow \left\|\frac{x}{2} + \frac{y}{2}\right\|^2 + \left\|\frac{x}{2} - \frac{y}{2}\right\|^2 = 2\left\|\frac{x}{2}\right\|^2 + 2\left\|\frac{y}{2}\right\|^2$$

$$\Rightarrow \left\|\frac{x}{2} - \frac{y}{2}\right\|^2 = 2\left\|\frac{x}{2}\right\|^2 + 2\left\|\frac{y}{2}\right\|^2 - \left\|\frac{x}{2} + \frac{y}{2}\right\|^2$$

$$\Rightarrow \left\|\frac{x-y}{2}\right\|^2 = 2\left\|\frac{x}{2}\right\|^2 + 2\left\|\frac{y}{2}\right\|^2 - \left\|\frac{x+y}{2}\right\|^2$$

$\Rightarrow \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 - 4d^2$

$= 4d^2 - 4d^2 = 0$

$\Rightarrow \|x-y\|^2 = 0$

$\Rightarrow \|x-y\| = 0$

$\Rightarrow x = y.$

This complete the proof of theorem

* Jordan-von Neumann Theorem:-

Statement:- A normed space is an inner product space if and only if the norm of the normed space satisfies the parallelogram law.

Proof:- Let X be a ~~linear~~ inner product space and an inner product $\langle \cdot, \cdot \rangle$ defined on X such that,

$\langle x, x \rangle = \|x\|^2$

then we have to show that parallelogram law satisfied.

Now

$$\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle$$

$$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle + \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle$$

$$= \|x\|^2 + \|y\|^2 + \|x\|^2 + \|y\|^2$$

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

Hence, parallelogram law satisfied

Conversely:- suppose that parallelogram law satisfies, i.e.

$\|x+y\|^2+\|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ — (1)

then we have to show that X is a inner product space.

for showing as such, we introduce polarisation identity as inner product i.e.

$$\langle x,y\rangle = \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right] \quad \text{— (ii)}$$

Here, we shall prove the following.

① $\langle x,x\rangle \geq 0$ and $\langle x,x\rangle = 0$ iff $x=0$

② $\langle x,y\rangle = \overline{\langle y,x\rangle}$ $\forall x,y \in X$.

③ $\langle \alpha x,y\rangle = \alpha\langle x,y\rangle$ $\forall \alpha \in K$

④ $\langle x+z,y\rangle = \langle x,y\rangle + \langle z,y\rangle$ $\forall x,y,z \in X$

Verification:-

① for all $x \in X$ we have from (ii)

$$\Rightarrow \langle x,x\rangle = \frac{1}{4}\left[\|x+x\|^2 - \|x-x\|^2 + i\|x+ix\|^2 - i\|x-ix\|^2\right]$$

$$\Rightarrow \frac{1}{4}\left[(2\|x\|)^2 + i\|(1+i)x\|^2 - i\|(1-i)x\|^2\right]$$

$$\Rightarrow \frac{1}{4}\left[4\|x\|^2 + i|1+i|^2\|x\|^2 - i|1-i|^2\|x\|^2\right]$$

$$\Rightarrow \frac{1}{4}\left[4\|x\|^2 + 2\|x\|^2 - 2\|x\|^2\right]$$

$$\Rightarrow \frac{1}{4}(4\|x\|^2) \Rightarrow \|x\|^2$$

i.e. $\langle x,x\rangle = \|x\|^2$

since $\|x\|^2 \geq 0$

$\therefore \boxed{\langle x,x\rangle \geq 0}$

and if $\|x\|^2 = 0 \Leftrightarrow x = 0$

$\therefore \boxed{\langle x,x\rangle = 0 \Leftrightarrow x = 0}$

This proves the first part.

② for all $x,y \in X$ we have from (ii)

$$\langle x,y\rangle = \frac{1}{4}\left[\|x+y\|^2 + \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right]$$

$$\therefore \overline{\langle y,x\rangle} = \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2 + i\|x-iy\|^2 - i\|x+iy\|^2\right]$$

$$= \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2 + i\|x-iy\|^2 - i\|x+iy\|^2\right]$$

$$= \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2 + i\|x-i(-iy)\|^2 - i\|x+i(-iy)\|^2\right]$$

$$= \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right]$$

$$= \langle x,y\rangle$$

$$= \frac{1}{4}\left[\|Ri+x\|^2 - i\|Ri-x\|^2 + i\|R-x\|^2 - \|R+x\|^2\right]$$

$$= \langle x, Ri\rangle$$

i.e. $\langle y, x\rangle = \overline{\langle x, y\rangle}$

$\langle Ry, x\rangle = \overline{\langle x, Ry\rangle} = \langle Rx, y\rangle$

i.e.

$$\langle x, Ry\rangle = \overline{\langle Ry, x\rangle}$$

(A) for all $x, y, z \in X$ we have from (ii)

$$\langle x+z, y\rangle = \frac{1}{4}\left[\|R(x+z)+y\|^2 + \|R(x+z)-y\|^2 - i\|R(x+z)+iy\|^2 - i\|R(x+z)-iy\|^2\right]$$

Now, ——— (C)

$$\Rightarrow \|x+y+z$$

$$\Rightarrow \|R(z+x)+z\|^2 - \|R(z+x)-z\|^2 =$$

$$2\left\|\frac{R}{2}(x+z)+\frac{z}{2}\right\|^2 - 2\left\|\frac{R}{2}(x+z)-\frac{z}{2}\right\|^2 =$$

$$2\|Rz+x\|^2 + 2\|R(x+z)\|^2 - 2\|Rz-x\|^2 - 2\|R(x-z)\|^2 =$$

$$2\left\|\frac{R}{2}(z+x)+\frac{z}{2}\right\|^2 + 2\left\|\frac{R}{2}(z+x)-\frac{z}{2}\right\|^2 - 2\left\|\frac{R}{2}(z-x)+\frac{z}{2}\right\|^2 - 2\left\|\frac{R}{2}(z-x)-\frac{z}{2}\right\|^2 =$$

$$2\left\|\frac{R}{2}(z+x)+\frac{z}{2}\right\|^2 + 2\left\|\frac{R}{2}(z+x)-\frac{z}{2}\right\|^2 - 2\left\|\frac{R}{2}(z-x)+\frac{z}{2}\right\|^2 - 2\left\|\frac{R}{2}(z-x)-\frac{z}{2}\right\|^2 =$$

$$= \frac{1}{4}\left[\|R(z+x)+z\|^2 + \|R(z+x)-z\|^2 - \|R(z-x)+z\|^2 - \|R(z-x)-z\|^2\right]$$

$$= \frac{1}{4}\left[\|Ri+x\|^2 - i\|Ri-x\|^2 + i\|R-x\|^2 - \|R+x\|^2\right]$$

$$= \langle x, Ry\rangle$$

i.e. $\langle y, x\rangle = \overline{\langle x, y\rangle}$

$\langle Ry, x\rangle = \overline{\langle x, Ry\rangle} = \langle Rx, y\rangle$

i.e.

$$\langle x, Ry\rangle = \overline{\langle Ry, x\rangle}$$

(A) for all $x, y, z \in X$ we have from (ii)

$$\langle x+z, y\rangle = \frac{1}{4}\left[\|R(x+z)+y\|^2 + \|R(x+z)-y\|^2 - i\|R(x+z)+iy\|^2 - i\|R(x+z)-iy\|^2\right]$$

Now, ——— (C)

$$\Rightarrow \|x+y+z$$

$$\Rightarrow \|R(z+x)+z\|^2 - \|R(z+x)-z\|^2 =$$

$$2\left\|\frac{R}{2}(z+x)+\frac{z}{2}\right\|^2 - 2\left\|\frac{R}{2}(z+x

$= 8\langle x, \frac{y}{2}\rangle + 8\langle z, \frac{y}{2}\rangle$ ——— (D)

from (C) and (D) we have,

$\langle x+z, y\rangle = \frac{1}{4}\left[8\langle x, \frac{y}{2}\rangle + 8\langle z, \frac{y}{2}\rangle\right]$

$\langle x+z, y\rangle = 2\langle x, \frac{y}{2}\rangle + 2\langle z, \frac{y}{2}\rangle$ ——— (E)

putting $z=0$ then

$\langle x, y\rangle = 2\langle x, y/2\rangle$

putting $x=0$ then

$\langle z, y\rangle = 2\langle z, \frac{y}{2}\rangle$

i.e. $\langle x, y\rangle = 2\langle x, \frac{y}{2}\rangle$

$\langle z, y\rangle = 2\langle z, \frac{y}{2}\rangle$

Thus

$\langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$

* **Orthogonal Sets:-** A non-empty set of an inner product space X is said to be orthogonal set $x \perp y$ $\forall$ $x, y \in S$ and $x \neq y$.

* **Orthonormal Set:-** Let S be a subset of an inner product space X then S is called an orthonormal set if

$\langle x_j, x_k\rangle = 0$ when $j \neq k$

and $\langle x_j, x_k\rangle = 1$ when $j = k$.

* **$S^\perp$ (S orthogonal):-** $S^\perp$ (S orthogonal) we means the set of all vector x such that $x \perp S$ i.e.

$S^\perp = \{x \in X : x \perp S\}$

* **Some basic properties of IPS**

Let X be an inner product space, λ, μ be any scalars and x, y, z are elements of X.

(a) $\langle x, \lambda y + \mu z\rangle = \bar{\lambda}\langle x, y\rangle + \bar{\mu}\langle x, z\rangle$

Proof:- we have,

$\langle x, \lambda y + \mu z\rangle = \overline{\langle \lambda y + \mu z, x\rangle}$

$= \overline{\lambda\langle y, x\rangle + \mu\langle z, x\rangle}$

$= \overline{\lambda\langle y, x\rangle} + \overline{\mu\langle z, x\rangle}$

$= \bar{\lambda}\langle y, x\rangle + \bar{\mu}\langle z, x\rangle$

$= \bar{\lambda}\langle x, y\rangle + \bar{\mu}\langle x, z\rangle$

Thus,

$\langle x, \lambda y + \mu z\rangle = \bar{\lambda}\langle x, y\rangle + \bar{\mu}\langle x, z\rangle$

(b) $\langle x - \lambda y, x - \lambda y\rangle = \|x\|^2 - 2\mathrm{Re}(\lambda\langle y, x\rangle) + |\lambda|^2 \|y\|^2$

Proof:- we have,

$\langle x - \lambda y, x - \lambda y\rangle = \langle x, x - \lambda y\rangle - \lambda\langle y, x - \lambda y\rangle$

$= \langle x,x\rangle - \bar{\lambda}\langle x,y\rangle - \lambda\langle y,x\rangle + \lambda\bar{\lambda}\langle y,y\rangle$

$= \|x\|^2 - \bar{\lambda}\langle x,y\rangle - \lambda\langle y,x\rangle + |\lambda|^2\|y\|^2$

$\{\because \|x\|^2 = \langle x,x\rangle,\ z\bar{z} = |z|^2\}$

$= \|x\|^2 - (\lambda\langle y,x\rangle + \bar{\lambda}\overline{\langle y,x\rangle}) + |\lambda|^2\|y\|^2$

$= \|x\|^2 - 2\mathrm{Re}(\lambda\langle y,x\rangle) + |\lambda|^2\|y\|^2$

$\{\because z+\bar{z} = 2\mathrm{Re}\,z\}$

Thus

$\langle x-\lambda y, x-\lambda y\rangle = \|x\|^2 - 2\mathrm{Re}(\lambda\langle y,x\rangle) + |\lambda|^2\|y\|^2$

10* The Triangle inequality :-

Statement :- for any two vectors x and y in an inner-product space X then,

$\|x+y\| \le \|x\| + \|y\|$

Proof :- given that $x, y \in X$ where X is an inner product space then

$\|x+y\|^2 = \langle x+y, x+y\rangle$

$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$

$= \langle x,x\rangle + \langle x,y\rangle + \overline{\langle x,y\rangle} + \langle y,y\rangle$

$= \|x\|^2 + 2\mathrm{Re}\langle x,y\rangle + \|y\|^2$

$\le \|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2$

$\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$

$\{$by cauchy Schwarz inequality$\}$

$\|x+y\|^2 \le (\|x\| + \|y\|)^2$

$\|x+y\| \le \|x\| + \|y\|$

This is called triangle inequality.

* Basic Properties of orthogonal vectors :-

(i) $x \perp y \Rightarrow y \perp x$

(ii) $x \perp y \Rightarrow x \perp (-y)$

(iii) $x \perp x \Leftrightarrow x = 0$

(iv) for any subset S, $S^\perp$ is called subspace of X.

(v) $X^\perp = \{0\}$, $\{0\}^\perp = X$

(vi) $S \cap S^\perp = \{0\}$

(vii) $S \subseteq T \Rightarrow S^\perp \supset T^\perp$

* Gram-Schmidt Orthogonalization Process

Statement :- Let $S = \{x_1, x_2, \ldots\}$ be a linearly independent sequence in an inner-product space. Then $\exists$ an orthonormal sequence $T = \{y_1, y_2, \ldots\}$ such that $L(S) = L(T)$

Proof :- since S is linearly independent, all x_k are non-zero vectors.

Define,

$y_1 = \frac{x_1}{\|x_1\|}$

so, that $\|y_1\| = 1$.

Define v,

$v = x_2 - (x_2, y_1)y_1$.

Then clearly $v \perp y_1$ and $v \neq 0$.

since $\{x_1, x_2\}$ is linearly independent.
Hence,

$$y_2 = \frac{v}{\|v\|}$$

is orthogonal to y_1 and $\|y_2\| = 1$.
In general we proceed inductively.
defining

$$v = x_n - \sum_{k=1}^{n-1} (x_n, y_k) y_k \text{ and } y_n = \frac{v}{\|v\|}$$

Then it is clear from the construction that S and T have the same linear span i.e.

$$L(S) = L(T)$$

Hence Proved.

15 * **Theorem:** In an inner product space, the inner product is jointly continuous.

Proof: - let X be an inner-product space and let $\{x_n\}$ and $\{y_n\}$ be sequences in X such that

$x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Then

$$|(x_n, y_n) - (x, y)| = |(x_n, y_n) - (x_n, y) + (x_n, y) - (x, y)|$$

$$= |(x_n, y_n - y) + (x_n - x, y)|$$

$$\leq |(x_n, y_n - y)| + |(x_n - x, y)|$$

$$\leq \|x_n\| \|y_n - y\| + \|x_n - x\| \|y\|$$

by Cauchy Schwarz inequality,
$\because \{x_n\}$ is convergent, it follows that $\|x_n\|$ is bounded, so let $\|x_n\| \leq m$ for $n = 1, 2, 3, \ldots$, Hence

$$|(x_n, y_n) - (x, y)| \leq m \|y_n - y\| + \|x_n - x\| \|y\|$$

$$\to 0 \text{ as } n \to \infty$$

because $\|x_n - x\| \to 0$, $\|y_n - y\| \to 0$ as $n \to \infty$

Thus,

$$\lim_{n \to \infty} (x_n, y_n) = (x, y)$$

This completes the proof.

17 * **Theorem:** - let M be a proper closed linear subspace of a Hilbert H. Then $\exists$ a nonzero vector z_0 in H such that $z_0 \perp M$.

Proof: - $\because$ M is a proper closed subspace of H. $\exists$ a vector x in H s.t. $x \notin M$. let

$$d = d(x, M) = \inf_{y \in M} \|x - y\|$$

since $x \notin M$, we have $d > 0$.
$\exists$ a vector unique vector y_0 in M of smallest norm s.t.

$$\|x - y_0\| = d.$$

let $z_0 = x - y_0$ then.

$\|z_0\| = \|x - y_0\| = d > 0$.

Hence, z_0 is a non-zero vector.

Then we have to prove that $z_0 \perp M$

we have to show that

$(z_0, y) = 0 \quad \forall\, y \in M$ and scalar α,

let

$z_0 - \alpha y = x - y_0 - \alpha y = x - (y_0 + \alpha y)$

Now, $y, y_0 \in M$, α is scalar $\Rightarrow y_0 + \alpha y \in M$.

we have,

$\|x - (y_0 + \alpha y)\| \geq d = \|z_0\|$

$\therefore \|z_0 - \alpha y\|^2 \geq \|z_0\|^2$

$\Rightarrow \|z_0 - \alpha y\|^2 - \|z_0\|^2 = (z_0 - \alpha y, z_0 - \alpha y) - (z_0, z_0) \geq$

$\Rightarrow (z_0, z_0) - \bar{\alpha}(z_0, y) - \alpha(y, z_0) + \alpha\bar{\alpha}(y, y) - (z_0, z_0) \geq 0$

$\Rightarrow -\bar{\alpha}(z_0, y) - \alpha(y, z_0) + \alpha\bar{\alpha}(y, y) \geq 0 \quad \text{---} (1)$

let $\alpha = \beta(z_0, y)$, where β is any arbitrary real number Then,

$\bar{\alpha} = \overline{\beta(z_0, y)}$

from eqn (1) we have,

$\Rightarrow -\overline{\beta(z_0, y)}(z_0, y) - \beta(z_0, y)\overline{(z_0, y)} + \beta^2 (z_0, y)\overline{(z_0, y)} \|y\|^2 \geq 0$

$\Rightarrow -2\beta |(z_0, y)|^2 + \beta^2 |(z_0, y)|^2 \|y\|^2 \geq 0$

Then we get,

$\beta |(z_0, y)|^2 \{\beta \|y\|^2 - 2\} \leq 0 \quad \text{---} (2)$

suppose $(z_0, y) \neq 0$ Since β is arbitrary choose β is positive s.t.

$\beta \|y\|^2 < 2$

from eqn (2) we get

$\beta |(z_0, y)|^2 \{\beta \|y\|^2 - 2\} < 0.$

which is contradiction of (2) Hence $(z_0, y) = 0$ showing that $z_0 \perp y$

Thus

$z_0 \perp y \quad \forall\, y \in M$

Therefore

$z_0 \perp M$

This completes the proof of theorem

15 *

Theorem :- If M and N are closed linear subspace of a Hilbert space H such that $M \perp N$, then the linear subspace $M + N$ is closed.

Proof :- To show that $M + N$ is closed, we have to prove that it contains all its limit points.

let z be a limit point of $M + N$. $\exists$ a sequence $\{z_n\}$ in $M + N$ s.t. $z_n \to z$ in H.

$\because M \perp N$, $M \cap N = \{0\}$ and $M + N$ is the direct sum of the subspaces M and N.

$\therefore z_n$ can be written uniquely as

$z_n = x_n + y_n$, $x_n \in M$ and $y_n \in N$.

taking two points $z_m = x_m + y_m$ and $z_n = x_n + y_n$ we have

$z_m - z_n = (x_m - x_n) + (y_m - y_n)$

$\because x_m, x_n \in M \Rightarrow x_m - x_n \in M$.

similarly, $y_m - y_n \in N$.

But $M \perp N$, it follows that,

$(x_m - x_n) \perp (y_m - y_n)$

Hence by Pythagorean theorem we get,

$\|(x_m - x_n) + (y_m - y_n)\|^2 = \|x_m - x_n\|^2 + \|y_m - y_n\|^2$

or $\|z_m - z_n\|^2 = \|x_m - x_n\|^2 + \|y_m - y_n\|^2$ ——— (1)

Since, $\{z_n\}$ is a convergent sequence in H.

it is a cauchy sequence in H.

Thus,

$\|z_m - z_n\|^2 \to 0$ as $m \to \infty$ ——— (2)

using (2) in (1) we get

$\|x_m - x_n\|^2 \to 0$ and $\|y_m - y_n\|^2 \to 0$ as $m, n \to \infty$

$\because \{x_n\}$ and $\{y_n\}$ are Cauchy sequence in M and N respectively.

$\Rightarrow$ H is complete and M and N are closed subspace of complete space $\therefore$ M and N are complete.

Now, $z = \lim_{n\to\infty} z_n = \lim_{n\to\infty} (x_n + y_n)$

$= \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$

$= x + y \in M + N$.

Thus, $z = x + y \in M + N$.

$\therefore M + N$ is closed.